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Two-dimensional vector invariant rings of Abelian p-groups ☆

Jianjun Chuai

Department of Mathematics & Statistics, Queen's University, Kingston, Ontario K7L 3N6, Canada

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Abstract

For any faithful representation V of a non-trivial p-group over a field of characteristic $p > 0$, it is known that the ring of vector invariants of m copies of V is not Cohen–Macaulay if $m \geqslant 3$. However, much less is known about the case $m = 2$. In this paper we show that, if $m = 2$ and the group is an Abelian p-group, then the ring of invariants of $2V$ is a complete intersection in some cases and is not Cohen–Macaulay in most cases. As a corollary we obtain that if the field is $\mathbb{F}_p$ and the ring of invariants of the representation V is a polynomial ring, then the ring of invariants of $2V$ is either a complete intersection or not Cohen–Macaulay.

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1. Introduction

We let V be a vector space of dimension n over a field $\mathbf{F}$ of characteristic $p \geqslant 0$ and let $\mathbf{F}[V]$ be the symmetric algebra of V^* (the dual of V). If $\{x_1, \ldots, x_n\}$ is a basis for V^*, then $\mathbf{F}[V]$ can be identified with the polynomial ring $\mathbf{F}[x_1, \ldots, x_n]$. Let $G \subseteq GL(V)$ be a finite group. Then the elements of G act on $\mathbf{F}[V]$ as algebra automorphisms and we form the subring $\mathbf{F}[V]^G$ of G-invariant polynomials.

It is well-known that $\mathbf{F}[V]^G$ is Cohen–Macaulay if p does not divide the order of G ([4], see [1, Section 6.4]), but $\mathbf{F}[V]^G$ often fails to be Cohen–Macaulay if p divides the order of G. Good examples of the latter case are given by vector invariants. Let mV be the direct sum of m copies of V and let G act on mV diagonally. Then $\mathbf{F}[mV]^G$ is called

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E-mail address: chuai@mast.queensu.ca.

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the (m-dimensional) ring of vector invariants (see [13]). If p divides $|G|$, Kemper proved in [5] that $\mathbf{F}[mV]^G$ is not Cohen–Macaulay for sufficiently large m. Furthermore, if G is a non-trivial p-group it is shown in [2] that $\mathbf{F}[mV]^G$ is not Cohen–Macaulay if $m \geqslant 3$. The purpose of this paper is to investigate the structure of the ring $\mathbf{F}[2V]^G$ when G is an Abelian p-group.

2. Preliminaries

In studying whether or not the two-dimensional vector invariant ring $\mathbf{F}[2V]^G$ is Cohen–Macaulay we need only to consider the case where $\mathbf{F}[V]^G$ is Cohen–Macaulay as the following result shows.

Proposition 2.1. *If* $\mathbf{F}[2V]^G$ *is Cohen–Macaulay, then* $\mathbf{F}[V]^G$ *is Cohen–Macaulay.*

Proof. Let $\mathbf{F}[2V] = \mathbf{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. It is easy to see that

$$\mathbf{F}[2V]^G = \mathbf{F}[x_1, \ldots, x_n]^G \oplus \big((y_1, \ldots, y_n)\mathbf{F}[2V]\big)^G.$$

If $\{h_1, \ldots, h_n\}$ is a homogeneous system of parameters for $\mathbf{F}[x_1, \ldots, x_n]^G$, then it is a partial homogeneous system of parameters for $\mathbf{F}[2V]^G$ and thus $\mathbf{F}[2V]^G$ is a free $A := \mathbf{F}[h_1, \ldots, h_n]$-module (of infinite rank) since $\mathbf{F}[2V]^G$ is Cohen–Macaulay by assumption. It follows that $\mathbf{F}[V]^G$ is a projective, hence free, A-module (see, for example, [12, Proposition 6.1.1]). Thus $\mathbf{F}[V]^G$ is Cohen–Macaulay. $\square$

In the rest of this paper we assume that $p > 0$. We know that for any p-group $P \subseteq GL(V)$ if $\mathbf{F}[2V]^P$ is Cohen–Macaulay then P can be generated by bireflections as a group acting on $2V$ (see [5, Corollary 3.7]), or equivalently, by reflections as a group acting on V. Recall that a reflection fixes point-wise a proper hyperplane in V, while a bireflection fixes point-wise a subspace of co-dimension 2 or 1. By this result, in order to study conditions under which $\mathbf{F}[2V]^P$ is or is not Cohen–Macaulay, we may assume that P is generated by reflections. If $\mathbf{F}$ is a finite field, the upper unipotent group U_n (namely, the group of upper triangular matrices with 1's along the diagonal) is a Sylow p-subgroup of $GL(V)$. So every p-subgroup of $GL(V)$ can be assumed to be a subgroup of U_n. Shank and Wehlau proved that $\mathbb{F}_p[2V]^{U_n}$ is not Cohen–Macaulay for $n > 2$ and $p > 2$ (see [8, Theorem 8.7]). In fact, they proved this theorem for $n = 3$, but by considering a stabilizer subgroup and using induction on n their result generalizes. So we have shown that for the biggest reflection p-group U_n, the ring $\mathbb{F}_p[2V]^{U_n}$ is not Cohen–Macaulay when $n > 2$ and $p > 2$. Of course, this group is not Abelian.

In the rest of this paper we will assume that the group G is a non-trivial Abelian p-group generated by reflections unless stated otherwise. In this setting, let us show first that under a certain carefully chosen basis of V the matrix of G takes a particular form. To this end, let $\{g_1, \ldots, g_r\}$ be a minimal set of generating reflections for G. Each g_i has order p (this is easy to see directly), so in fact G is an elementary Abelian p-group.

Definition 2.2. By definition, we have for each reflection g_i a *root vector* v_i, that is, a vector with the property that $(g_i - 1)(V) = \mathbf{F}v_i$. We note that $v_i \in V^G$ (see [9, Proposition 2.2]).
We define the root space of V, denoted $\mathcal{R}(G, V)$, to be $\sum_{g\in G}(g-1)V$.

To explain the above definition of root space, note that from the identity

$$(g_i g_j - 1) = (g_i - 1)(g_j - 1) + (g_i - 1) + (g_j - 1)$$

applied to vectors of V and the fact that $v_i \in V^G$ we see that $\mathcal{R}(G, V)$ is in fact spanned by $\{v_1, \dots, v_r\}$. In particular we have $\mathcal{R}(G, V) \subseteq V^G$.

For the Abelian reflection p-group G, let $\mathrm{d}(G, V)$ denote $\dim(\mathcal{R}(G, V))$ and $\mathrm{r}(G)$ denote the rank of G. Of course we have $\mathrm{d}(G, V) \leqslant \mathrm{r}(G)$. Now assume $\mathrm{d}(G, V) = d$ and let $\{v_1, \dots, v_d\}$ be linearly independent. Furthermore, let $\dim(V^G) = s$. We can choose a basis $\{u_1, \dots, u_n\}$ for V such that $u_i = v_i$ for $1 \leqslant i \leqslant d$ and $\{u_1, \dots, u_s\}$ is a basis for V^G. Then under this basis each element g of G has the following form of matrix representation:

$$g = \begin{pmatrix} I_d & 0 & A \\ 0 & I_{s-d} & 0 \\ 0 & 0 & I_{n-s} \end{pmatrix}.$$

In what follows the above matrix representation of G will play an important role in the proofs of the results.

We have seen that $\mathrm{d}(G, V) \leqslant \mathrm{r}(G)$. The analysis of the structure of $\mathbf{F}[2V]^G$ depends entirely on whether or not $\mathrm{d}(G, V)$ equals $\mathrm{r}(G)$. In particular, we can prove that $\mathbf{F}[2V]^G$ is a complete intersection when the root space has maximal dimension. Further, Ian Hughes has conjectured

Conjecture. *If* $\mathbf{F} = \mathbb{F}_p$ *and* $\mathrm{d}(G, V) < \mathrm{r}(G)$, *then* $\mathbf{F}[2V]^G$ *is not Cohen–Macaulay.*

We are able to prove the conjecture is true in an important special case.

3. The case $\mathbf{d}(G, V) = \mathbf{r}(G)$

Let $P \subseteq GL(V)$ be a p-group. Then P is called a Nakajima-group if there is a basis $\{x_1, \dots, x_n\}$ of V^* such that under this basis P is upper triangular and such that $P = P_1 P_2 \cdots P_n$, where

$$P_i = \{g \in G \mid g x_j = x_j \text{ for } j \neq i\}.$$

Clearly, every Nakajima-group is a reflection group. Further, the invariant ring of each Nakajima group is a polynomial ring generated by the orbit-products of the x_i's (see [7]). In the case that $\mathbf{F} = \mathbb{F}_p$, P is a Nakajima-group if and only if $\mathbf{F}[V]^P$ is a polynomial ring (see [10], or [3, Theorem 2.1]).

Recall that a finitely generated graded algebra A of Krull dimension m is called a complete intersection if there is a polynomial algebra R in $m+s$ variables and a homogeneous ideal $I \lhd R$ generated by s elements such that $R/I \cong A$.

We now come to the proof of the following theorem.

Theorem 3.1. *Assume that G is an Abelian p-group generated by reflections $g_1, \ldots, g_r$, and assume that under a basis $\{v_1, \ldots, v_n\}$ of V with V^G being spanned by $\{v_1, \ldots, v_s\}$,*

$$(g_i - 1)(v_j) = a_{ij} v_i, \quad a_{ij} \in \mathbf{F},$$

for $1 \leqslant i \leqslant r$ and $s+1 \leqslant j \leqslant n$. Let $\{x_1, \ldots, x_n\}$ be the dual basis of $\{v_1, \ldots, v_n\}$ and let

$$\mathbf{F}[2V] = \mathbf{F}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

Define

$$\begin{aligned} \ell_i(x) &:= a_{is+1}x_{s+1} + \cdots + a_{in}x_n, \\ \ell_i(y) &:= a_{is+1}y_{s+1} + \cdots + a_{in}y_n, \\ h_i &:= \ell_i(x)y_i - x_i\ell_i(y), \\ N(x_i) &:= x_i^p - x_i\ell_i(x)^{p-1}, \\ N(y_i) &:= y_i^p - y_i\ell_i(y)^{p-1} \end{aligned}$$

for $1 \leqslant i \leqslant r$. Further, let

$$A_r = \mathbf{F}\big[x_i, y_i, N(x_j), N(y_j) \mid r+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant r\big].$$

Then

$$\mathbf{F}[V]^G = \mathbf{F}\big[x_i, N(x_j) \mid r+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant r\big]$$

is a polynomial ring and

$$\mathbf{F}[2V]^G = \bigoplus_{0 \leqslant i_j < p} A_r h_1^{i_1} \cdots h_r^{i_r}$$

is a complete intersection.

Proof. First, note that for each i, $\ell_i(x), \ell_i(y), h_i, N(x_i), N(y_i)$ are all invariants.

It is clear that G is a Nakajima group with respect to the basis $\{x_1, \ldots, x_n\}$, so the first result follows immediately. It is easy to see that

$$\big\{x_i, y_i, N(x_j), N(y_j) \mid r+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant r\big\}$$

is a homogeneous system of parameters for $\mathbf{F}[2V]^G$ and thus A_r is a polynomial ring. We prove the second result by induction on r.

Let $r = 1$. Without loss of generality we may assume $a_{1s+1} \neq 0$. Then $\{x_1, \ldots, x_s, \ell_1(x), x_{s+2}, \ldots, x_n\}$ is a basis for V^*, $(g_1 - 1)x_1 = -\ell_1(x)$ and g_1 fixes the other elements of this basis. Let

$$B_1 = \mathbf{F}\big[x_2, \ldots, x_s, \ell_1(x), x_{s+2}, \ldots, x_n, y_2, \ldots, y_s, \ell_1(y), y_{s+2}, \ldots, y_n, N(x_1), N(y_1)\big].$$

Then by [11, Proposition 11],

$$\mathbf{F}[2V]^G = \bigoplus_{0 \leqslant i < p} B_1 h_1^i.$$

But clearly we have $B_1 = A_1$. So the result follows in this case.

Now let $r > 1$ and assume

$$\mathbf{F}[2V]^{\langle g_1, \ldots, g_{r-1}\rangle} = \bigoplus_{0 \leqslant i_j < p} A_{r-1} h_1^{i_1} \cdots h_{r-1}^{i_{r-1}},$$

where

$$A_{r-1} = \mathbf{F}\big[x_i, y_i, N(x_j), N(y_j) \mid r \leqslant i \leqslant n,\ 1 \leqslant j \leqslant r-1\big].$$

Then we have

$$\mathbf{F}[2V]^G = \bigoplus_{0 \leqslant i_j < p} A_{r-1}^{g_r} h_1^{i_1} \cdots h_{r-1}^{i_{r-1}}.$$

Moreover,

$$A_{r-1}^{g_r} = \mathbf{F}[x_i, y_i \mid r \leqslant i \leqslant n]^{g_r} \otimes_{\mathbf{F}} \mathbf{F}\big[N(x_j), N(y_j) \mid 1 \leqslant j \leqslant r-1\big].$$

But by the proof of the case $r = 1$ we have

$$\mathbf{F}\big[x_i, y_i \mid r \leqslant i \leqslant n\big]^{g_r} = \bigoplus_{0 \leqslant i < p} \mathbf{F}\big[x_i, y_i, N(x_r), N(y_r) \mid r+1 \leqslant i \leqslant n\big] h_r^i.$$

So we get

$$\mathbf{F}[2V]^G = \bigoplus_{0 \leqslant i_j < p} A_r h_1^{i_1} \cdots h_r^{i_r}.$$

Now each h_i satisfies a degree p monic polynomial over A_r:

$$h_i^p - \big(\ell_i(x)\ell_i(y)\big)^{p-1} h_i + \ell_i(y)^p N(x_i) - \ell_i(x)^p N(y_i) = 0,$$

we see that $\mathbf{F}[2V]^G$ is a complete intersection. $\square$

Corollary 3.2. *Let G be an Abelian reflection p-group. If $\mathrm{d}(G,V)=\mathrm{r}(G)$, then $\mathbf{F}[V]^G$ is a polynomial ring and $\mathbf{F}[2V]^G$ is a complete intersection.*

Proof. Let $\mathrm{r}(G)=r$ and let $g_1,\dots,g_r$ be a minimal generating set of G with $(g_i-1)V=\mathbf{F}v_i$. Since $\mathrm{d}(G,V)=\mathrm{r}(G)$, $\{v_1,\dots,v_r\}$ is a linearly independent set. Thus there is a basis of V which extends $\{v_1,\dots,v_r\}$ and contains a basis of V^G. By Theorem 3.1 the results follow. $\square$

Example 3.3. Assume that under a basis of V,

$$G=\left\{\begin{pmatrix}1 & & & \alpha_1\\ & 1 & & \vdots\\ & & \ddots & \alpha_{n-1}\\ & & & 1\end{pmatrix}\;\middle|\;\alpha_i\in\mathbb{F}_p\right\}.$$

Then $\mathbf{F}[2V]^G$ is a complete intersection.

4. The case $\mathbf{d}(G,V)<\mathbf{r}(G)$

We recall the

Conjecture (Ian Hughes). *If $\mathbf{F}=\mathbb{F}_p$ and $\mathrm{d}(G,V)<\mathrm{r}(G)$, then $\mathbf{F}[2V]^G$ is not Cohen–Macaulay.*

In this section we prove the above conjecture in an important case. For any finite group N, any field $\mathbf{k}$, and any $\mathbf{k}N$-module M let $H^i(N,M)$ denote the ith cohomology group of N with coefficients in M. In the case where U is a finite dimensional vector space over $\mathbf{k}$ and $N\subseteq GL(U)$ is a finite group, there is a natural way to view $H^i(N,\mathbf{k}[U])$ as a $\mathbf{k}[U]^N$-module. We will need the following result (see [5, Corollary 1.6]).

Lemma 4.1. *Assume that $H^i(N,\mathbf{k}[U])=0$ for $1\leqslant i<j$, where $j>0$, and $0\neq\phi\in H^j(N,\mathbf{k}[U])$. Then*

$$\operatorname{depth}_{\mathrm{Ann}_{\mathbf{k}[U]^N}(\phi)}\big(\mathbf{k}[U]^N\big)=\min\big\{j+1,\operatorname{height}\big(\mathrm{Ann}_{\mathbf{k}[U]^N}(\phi)\big)\big\}.$$

In particular, $\mathbf{k}[U]^N$ is not Cohen–Macaulay if $\operatorname{height}(\mathrm{Ann}_{\mathbf{k}[U]^N}(\phi))>j+1$.

Note that if N is a non-trivial p-group and $\mathbf{k}$ has characteristic p, there are non-trivial group homomorphisms from N to $\mathbf{k}^+$ and these homomorphisms can be viewed as non-zero elements of $H^1(N,\mathbf{k})$ and hence of $H^1(N,\mathbf{k}[U])$. In particular, $H^1(N,\mathbf{k}[U])\neq 0$ in this case.

If X is a subset of U, denote by $\mathcal{I}_{\mathbf{k}[U]^N}(X)$ the vanishing ideal of $\mathbf{k}[U]^N$ on X. We have the following result (see [5, Proposition 3.5]).

Lemma 4.2. *Assume that* $\mathbf{k}$ *is an algebraically closed field and let* $\phi \in H^1(N, \mathbf{k})$ *be nonzero with kernel* $H \lhd N$. *Then for* $I = \mathrm{Ann}_{\mathbf{k}[U]^N}(\phi)$ *we have*

$$\sqrt{I} = \bigcap_{g \in N \setminus H} \mathcal{I}_{\mathbf{k}[U]^N}\big(U^g\big).$$

Theorem 4.3. *Let* $\mathbf{F} = \mathbb{F}_p$ *and assume that* G *is an Abelian reflection* p*-group. If there exist reflections* $g, h \in G$ *such that* $V^g \neq V^h$ *and* gh *is a reflection, then* $\mathbf{F}[2V]^G$ *is not Cohen–Macaulay.*

Proof. Since $V^g \neq V^h$, $g^i h^j = 1$ if and only if $i \equiv j \equiv 0 \pmod p$. So g and h are linearly independent when viewing G as a vector space over $\mathbb{F}_p$. Thus $\{g, h\}$ is contained in a minimal generating set of G consisting of reflections. Let $\mathrm{r}(G) = r$ and assume that $\{g_1, \ldots, g_r\}$ is such a set with $g_1 = g$ and $g_2 = h$.

Let $\mathrm{d}(G, V) = d$ and let $\dim(V^G) = s$ (thus $d \leqslant s$). As mentioned early there is a basis $\{v_1, \ldots, v_n\}$ of V such that under this basis G has the following matrix representation:

$$\left\{ \begin{pmatrix} I_d & 0 & A_g \\ 0 & I_{s-d} & 0 \\ 0 & 0 & I_{n-s} \end{pmatrix} \,\middle|\, g \in G \right\}.$$

Since $\mathrm{rank}(A_{g_i}) = 1$, we can write $A_{g_i} = \beta_i \cdot \alpha_i$, where $\beta_i \in \mathbf{F}^d$ (column vectors) and $\alpha_i \in \mathbf{F}^{n-s}$ (row vectors). Let $\{x_1, \ldots, x_n\}$ be the dual basis of $\{v_1, \ldots, v_n\}$ and let $\mathbf{F}[2V] = \mathbf{F}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Then it is easy to see that the linear form $\ell_i := \alpha_i \cdot (x_{s+1}, \ldots, x_n)^t$ defines the hyperplane V^{g_i}. Since $V^{g_1} \neq V^{g_2}$, α_1 and α_2 are linearly independent. Since $g_1 g_2$ is a reflection, we can write $A_{g_1 g_2} = \beta \cdot \alpha$. Now from $\beta \cdot \alpha = \beta_1 \cdot \alpha_1 + \beta_2 \cdot \alpha_2$ and the fact that α_1 and α_2 are linearly independent we can see easily that $\beta_1, \beta_2 \in \mathbf{F}\beta$. Thus the matrix (β_1, β_2) has rank 1 and therefore we can find $B \in GL_d(\mathbf{F})$ such that both vectors $B \cdot \beta_1$ and $B \cdot \beta_2$ have all entries zeros except the first ones. We may assume $B \cdot \beta_1 = (1, 0, \ldots, 0)^t$ and $B \cdot \beta_2 = (b, 0, \ldots, 0)^t$ with $b \neq 0$. Since α_1 and α_2 are linearly independent, there exists $C \in GL_{n-s}(\mathbf{F})$ such that $\alpha_1 \cdot C = (1, 0, \ldots, 0)$ and $\alpha_2 \cdot C = (0, b^{-1}, 0, \ldots, 0)$. Note that

$$\begin{pmatrix} B & 0 & 0 \\ 0 & I_{s-d} & 0 \\ 0 & 0 & C^{-1} \end{pmatrix} \begin{pmatrix} I_d & 0 & A \\ 0 & I_{s-d} & 0 \\ 0 & 0 & I_{n-s} \end{pmatrix} \begin{pmatrix} B^{-1} & 0 & 0 \\ 0 & I_{s-d} & 0 \\ 0 & 0 & C \end{pmatrix} = \begin{pmatrix} I_d & 0 & BAC \\ 0 & I_{s-d} & 0 \\ 0 & 0 & I_{n-s} \end{pmatrix}.$$

So we may assume that under the basis $\{v_1, \ldots, v_n\}$,

$$A_{g_1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad A_{g_2} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\mathbf{k}$ be an algebraic closure of $\mathbf{F}$ (=$\mathbb{F}_p$) and $\overline{V} = \mathbf{k} \otimes_{\mathbf{F}} V$. To show that $\mathbf{F}[2V]^G$ is not Cohen–Macaulay we need only to show that $\mathbf{k}[\overline{V}]^G$ is not Cohen–Macaulay (see the proof of [5, Theorem 2.3]). For simplicity we still use $\mathbf{F}$ to denote $\mathbf{k}$ and use V to denote $\overline{V}$. We now prove the theorem by using induction on $t := n - \dim(V^G) = n - s$. First assume $t = 2$.

For any $g \in G$, let

$$A_g = \begin{pmatrix} \alpha_g & \beta_g \\ * & * \\ \vdots & \vdots \end{pmatrix}.$$

Define the group homomorphisms

$$\phi, \psi : G \to \mathbb{F}_p^+ \subset \mathbf{F}^+$$

by $\phi(g) = \alpha_g$ and $\psi(g) = \beta_g$ and let $N = \ker(\phi)$. Then for $I = \mathrm{Ann}_{\mathbf{F}[2V]^G}(\phi)$,

$$\sqrt{I} = \bigcap_{g \in G \setminus N} \mathcal{I}_{\mathbf{F}[2V]^G}\big((2V)^g\big)$$

by Lemma 4.2. We have $\alpha_g x_{n-1} + \beta_g x_n, \alpha_g y_{n-1} + \beta_g y_n \in \mathcal{I}_{\mathbf{F}[2V]^G}((2V)^g)$ and $\alpha_g \neq 0$ for $g \in G \setminus N$. Thus we have $f := y_{n-1}(x_{n-1}^{p-1} - x_n^{p-1}) \in \sqrt{I}$. We assert that $f \cdot \phi \neq 0$. Otherwise, there would exist an $h \in \mathbf{F}[2V]^N \subseteq \mathbf{F}[2V]^{\langle g_2 \rangle}$ such that $(g_1 - 1)h = f$. By [11, Proposition 11] we know that

$$\mathbf{F}[2V]^{\langle g_2 \rangle} = \mathbf{F}\big[x_2, \ldots, x_n, y_2, \ldots, y_n, x_1^p - x_1 x_n^{p-1}, y_1^p - y_1 y_n^{p-1}, x_1 y_n - x_n y_1\big].$$

By comparing y-degrees we may assume $h = f_1 \cdot (x_1 y_n - x_n y_1)$ with $f_1 \in \mathbf{F}[x_2, \ldots, x_n]$. But clearly $(g_1 - 1)h = f_1 \cdot (x_n y_{n-1} - x_{n-1} y_n) \neq f$ because y_{n-1} does not divide $f_1 \cdot (x_n y_{n-1} - x_{n-1} y_n)$. So $f \cdot \phi \neq 0$. Now from

$$\begin{aligned} (g-1)\big(y_1\big(x_n^p - x_n x_{n-1}^{p-1}\big)\big) &= -(\alpha_g y_{n-1} + \beta_g y_n)\big(x_n^p - x_n x_{n-1}^{p-1}\big) \\ &= x_n f \cdot \phi(g) - y_n\big(x_n^p - x_n x_{n-1}^{p-1}\big) \cdot \psi(g) \\ &= \big(x_n f \cdot \phi - y_n\big(x_n^p - x_n x_{n-1}^{p-1}\big) \cdot \psi\big)(g) \end{aligned}$$

we have

$$x_n f \cdot \phi - y_n\big(x_n^p - x_n x_{n-1}^{p-1}\big) \cdot \psi = 0.$$

Furthermore,

$$\begin{aligned}(g-1)\big(x_1x_{n-1}^{p-1}-x_1^p\big) &= -(\alpha_g x_{n-1}+\beta_g x_n)x_{n-1}^{p-1}+\big(\alpha_g x_{n-1}^p+\beta_g x_n^p\big)\\ &= \beta_g\big(x_n^p-x_nx_{n-1}^{p-1}\big)\\ &= \big(\big(x_n^p-x_nx_{n-1}^{p-1}\big)\cdot\psi\big)(g).\end{aligned}$$

So we have $(x_n^p-x_nx_{n-1}^{p-1})\cdot\psi=0$ and thus $x_n\in \mathrm{Ann}_{\mathbf{F}[2V]^G}(f\cdot\phi):=J$.

It is easy to see that $x_{n-1}^p-x_{n-1}x_n^{p-1}$, $y_{n-1}^p-y_{n-1}y_n^{p-1}\in\sqrt{I}\subseteq\sqrt{J}$ and that $x_{n-1}^p-x_{n-1}x_n^{p-1}$, $y_{n-1}^p-y_{n-1}y_n^{p-1}$, x_n form a partial system of parameters. So

$$\mathrm{height}(J)=\mathrm{height}\big(\sqrt{J}\big)\geqslant 3$$

and by Lemma 4.1, $\mathbf{F}[2V]^G$ is not Cohen–Macaulay.

Now assume $t>2$. Then we consider the stabilizer subgroup G_{v_n} of G. Clearly, $\{g_1,g_2\}\subset G_{v_n}$. If G_{v_n} can not be generated by reflections, then $\mathbf{F}[2V]^{G_{v_n}}$ is not Cohen–Macaulay, and thus $\mathbf{F}[2V]^G$ is not also (see [6, Theorem A]). If G_{v_n} is generated by reflections, then since $n-\dim(V^{G_{v_n}})<t$ and G_{v_n} satisfies the assumptions of the theorem, by induction $\mathbf{F}[2V]^{G_{v_n}}$ is not Cohen–Macaulay. It follows from [6, Theorem A] again that $\mathbf{F}[2V]^G$ is not Cohen–Macaulay. □

In the case that G is an Abelian Nakajima-group we have the following result.

Corollary 4.4. *Let $\mathbf{F}=\mathbb{F}_p$ and let G be an Abelian Nakajima p-group. Assume that $\mathrm{d}(G,V)<\mathrm{r}(G)$. Then $\mathbf{F}[2V]^G$ is not Cohen–Macaulay.*

Proof. Since G is a Nakajima group, there is a basis $\{x_1,\dots,x_n\}$ of V^* under which G is upper triangular and

$$G=G_1G_2\cdots G_n,$$

where

$$G_i=\{g\in G\mid gx_j=x_j\ \text{for}\ j\neq i\}.$$

Let $\{v_1,\dots,v_n\}\subseteq V$ be the basis dual to $\{x_1,\dots,x_n\}$. Assume $G_{i_j}\neq\{1\}$ for $1\leqslant j\leqslant s$ and $1\leqslant i_1<i_2<\cdots<i_s\leqslant n$. Then $G=G_{i_1}\times\cdots\times G_{i_s}$. If $\mathrm{rank}(G_{i_j})=1$ for all j, then $s=\mathrm{r}(G)$. Thus $\mathcal{R}(G,V)=\mathrm{Span}\{v_{i_1},\dots,v_{i_s}\}$ has dimension $\mathrm{r}(G)$, a contradiction. So there exists a j such that $\mathrm{rank}(G_{i_j})>1$. It follows that there are reflections $g,h\in G_{i_j}$ with $V^g\neq V^h$ and gh a reflection. By Theorem 4.3 the result follows. □

We have mentioned earlier that if $\mathbf{F}=\mathbb{F}_p$ and $P\subseteq GL(V)$ is a p-group, P is a Nakajima-group if and only if $\mathbf{F}[V]^P$ is a polynomial ring. Now combining Corollary 3.2 with Corollary 4.4 we have

Corollary 4.5. *Let $\mathbf{F} = \mathbb{F}_p$ and assume that $G \subseteq GL(V)$ is an Abelian p-group such that $\mathbf{F}[V]^G$ is a polynomial ring. Then $\mathbf{F}[2V]^G$ is either a complete intersection or not a Cohen–Macaulay ring. More precisely, if $\mathrm{d}(G,V) = \mathrm{r}(G)$, then $\mathbf{F}[2V]^G$ is a complete intersection, and if $\mathrm{d}(G,V) < \mathrm{r}(G)$, then $\mathbf{F}[2V]^G$ is not Cohen–Macaulay.*

The following example illustrates the above corollary.

Example 4.6. Let $\mathbf{F} = \mathbb{F}_p$ and assume that under a basis of V the group G takes the following form:

$$\left\{ \begin{pmatrix} I_d & A \\ 0 & I_{n-d} \end{pmatrix} \,\middle|\, A \in M_{d\times(n-d)}(\mathbf{F}) \right\}.$$

Then $\mathbf{F}[2V]^G$ is a complete intersection if $n = d+1$ and not Cohen–Macaulay if $n > d+1$.

5. An example

We know that for a p-group $G \subseteq GL(V)$, if $\mathbf{F} = \mathbb{F}_p$, then $\mathbf{F}[V]^G$ is a polynomial ring if and only if G is a Nakajima group. But this result does not extend to representations of p-groups over bigger fields: there are p-groups with polynomial invariants which are not Nakajima groups. One example of such a group is due to Stong (see [7, Example 4.5]). Our interest here is to study the structure of the two-dimensional vector invariant ring of this group. I am grateful to Eddy Campbell for suggesting I study this example.

Example 5.1. Here we consider $\mathbf{k}[2V]^G$, where $\mathbf{k} = \mathbb{F}_{p^3}$, $V^* = \langle x_1, x_2, x_3\rangle$, G acts on V^* and, with respect to the basis $\{x_1, x_2, x_3\}$,

$$G = \left\{ \begin{pmatrix} 1 & \alpha + \gamma v & \beta + \gamma w \\ & 1 & 0 \\ & & 1 \end{pmatrix} \,\middle|\, \alpha, \beta, \gamma \in \mathbb{F}_p \right\},$$

where $\{1, v, w\}$ is a basis for $\mathbf{k}$ over $\mathbb{F}_p$. Then $\mathbf{k}[2V]^G$ is Cohen–Macaulay.

Proof. Let $\mathbf{k}[2V] = \mathbf{k}[x_1, x_2, x_3, y_1, y_2, y_3]$. We denote by H the subgroup of G consisting of those elements with $\gamma = 0$, and by σ the element of G with $\alpha = \beta = 0$ and $\gamma = 1$. Let $L = \langle\sigma\rangle$. Then $G = H \times L$, and by Theorem 3.1,

$$\begin{aligned}
&\mathbf{k}[2V]^H \\
&\quad = \bigoplus_{0\leqslant i,j<p} \mathbf{k}\big[x_1, y_1, x_2^p - x_2x_1^{p-1}, x_3^p - x_3x_1^{p-1}, y_2^p - y_2y_1^{p-1}, y_3^p - y_3y_1^{p-1}\big]h_1^i h_2^j \\
&\quad = \bigoplus_{0\leqslant i_k<p} \mathbf{k}\big[x_1^p, y_1^p, x_2^p - x_2x_1^{p-1}, x_3^p - x_3x_1^{p-1}, y_2^p - y_2y_1^{p-1}, y_3^p - y_3y_1^{p-1}\big] \\
&\qquad\qquad \times x_1^{i_1} y_1^{i_2} h_1^{i_3} h_2^{i_4},
\end{aligned}$$

where $h_1 = x_1y_2 - x_2y_1$ and $h_2 = x_1y_3 - x_3y_1$. Note that x_1, y_1, h_1, h_2 are all G-invariants and that

$$\begin{aligned}
(\sigma-1)\bigl(x_2^p - x_2x_1^{p-1}\bigr) &= \bigl(v^p - v\bigr)x_1^p,\\
(\sigma-1)\bigl(x_3^p - x_3x_1^{p-1}\bigr) &= \bigl(w^p - w\bigr)x_1^p,\\
(\sigma-1)\bigl(y_2^p - y_2y_1^{p-1}\bigr) &= \bigl(v^p - v\bigr)y_1^p,\\
(\sigma-1)\bigl(y_3^p - y_3y_1^{p-1}\bigr) &= \bigl(w^p - w\bigr)y_1^p,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathbf{k}[2V]^G = \bigoplus_{0\leqslant i_k<p} &\mathbf{k}\bigl[x_1^p, y_1^p, x_2^p - x_2x_1^{p-1}, x_3^p - x_3x_1^{p-1}, y_2^p - y_2y_1^{p-1}, y_3^p - y_3y_1^{p-1}\bigr]^L\\
&\times x_1^{i_1}y_1^{i_2}h_1^{i_3}h_2^{i_4}.
\end{aligned}$$

Let

$$\begin{aligned}
u_1 &= \bigl(x_2^p - x_2x_1^{p-1}\bigr)^p - \bigl(v^p - v\bigr)^{p-1}\bigl(x_2^p - x_2x_1^{p-1}\bigr)x_1^{p(p-1)},\\
u_2 &= \bigl(y_2^p - y_2y_1^{p-1}\bigr)^p - \bigl(v^p - v\bigr)^{p-1}\bigl(y_2^p - y_2y_1^{p-1}\bigr)y_1^{p(p-1)},\\
u_3 &= \bigl(w^p - w\bigr)\bigl(x_2^p - x_2x_1^{p-1}\bigr) - \bigl(v^p - v\bigr)\bigl(x_3^p - x_3x_1^{p-1}\bigr),\\
u_4 &= \bigl(w^p - w\bigr)\bigl(y_2^p - y_2y_1^{p-1}\bigr) - \bigl(v^p - v\bigr)\bigl(y_3^p - y_3y_1^{p-1}\bigr),\\
h_3 &= \bigl(v^p - v\bigr)x_1^p\bigl(y_2^p - y_2y_1^{p-1}\bigr) - \bigl(v^p - v\bigr)y_1^p\bigl(x_2^p - x_2x_1^{p-1}\bigr).
\end{aligned}$$

Then they are all G-invariants. We have

$$\begin{aligned}
&\mathbf{k}\bigl[x_1^p, y_1^p, x_2^p - x_2x_1^{p-1}, x_3^p - x_3x_1^{p-1}, y_2^p - y_2y_1^{p-1}, y_3^p - y_3y_1^{p-1}\bigr]^L\\
&\quad= \mathbf{k}\bigl[\bigl(v^p - v\bigr)x_1^p, \bigl(v^p - v\bigr)y_1^p, x_2^p - x_2x_1^{p-1}, y_2^p - y_2y_1^{p-1}\bigr]^L \otimes_{\mathbf{k}} \mathbf{k}[u_3, u_4]\\
&\quad= \left(\bigoplus_{0\leqslant i<p} \mathbf{k}\bigl[x_1^p, y_1^p, u_1, u_2\bigr]h_3^i\right) \otimes_{\mathbf{k}} \mathbf{k}[u_3, u_4] \quad \text{(by [11, Proposition 11])}\\
&\quad= \bigoplus_{0\leqslant i<p} \mathbf{k}\bigl[x_1^p, y_1^p, u_1, u_2, u_3, u_4\bigr]h_3^i.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{k}[2V]^G &= \bigoplus_{0\leqslant i_k<p} \mathbf{k}\bigl[x_1^p, y_1^p, u_1, u_2, u_3, u_4\bigr]x_1^{i_1}y_1^{i_2}h_1^{i_3}h_2^{i_4}h_3^{i_5}\\
&= \bigoplus_{0\leqslant i,j,k<p} \mathbf{k}[x_1, y_1, u_1, u_2, u_3, u_4]h_1^ih_2^jh_3^k.
\end{aligned}$$

It is clear that $\{x_1, y_1, u_1, u_2, u_3, u_4\}$ is a homogeneous system of parameters for $\mathbf{k}[2V]^G$, therefore $\mathbf{k}[2V]^G$ is Cohen–Macaulay. $\square$

Example 5.1 shows that Hughes' Conjecture does not extend to bigger fields since, in this example, $\mathrm{d}(G, V) = 2 < \mathrm{r}(G) = 3$ and $\mathbf{k}[2V]^G$ is a Cohen–Macaulay ring. This example also shows that the second part of Corollary 4.5 does not hold if $\mathbf{F} \neq \mathbb{F}_p$.

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